MIXED CHARACTERISTIC PROBLEM FOR MOTION OF A GRANULAR AND ADDITIVELY COMPRESSIBLE MEDIUM

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The articles [1, 2] hold a prominent position among the large number of publications on problems of motion of granular media. In [1] the relations were formulated between the stresses and the strain rates on the basis of the concept of a plastic potential; they were applied in [2] to formulate mixed boundary-value problems and to solve problems on pressing a die or a lubricated wedge into a dilatating mass of soil. The present article can be regarded as a further development of [2] for a model of a body [3] which describes the motion of a noncompact medium loosening up when shifted and volume-compressible due to hydrostatic pressure.

§1. A granular medium is considered in which a planar irreversible deformation takes place under the condition of Coulomb limit state,

$$\sigma\sin\varphi+\tau_*=c\cos\varphi,\quad \tau_*=\left\{\frac{1}{4}(\sigma_x-\sigma_y)^2+\tau_{xy}^2\right\}^{\frac{1}{2}},\quad \sigma=\frac{1}{2}(\sigma_x+\sigma_y),$$

where σ_x, σ_y, and τ_{xy} are stress components; c and φ are constants of the medium.

It is assumed that during its flow the granular medium suffers a plastic change of the volume of the particles e_{ii} depending on the level of the hydrostatic part of the stressed state according to the rule $e_{ii}=\psi(\sigma)$.

The relation between the stresses and the strain rates in a granular body are assumed to be in the form of a generalized flow rule [1] as proposed in [3] and are based on the additivity of the increments of the volume plastic strains due to shear and to changes in hydrostatic pressure:

$$\begin{gathered}\varepsilon_x=\frac{\lambda}{2}\left(\sin\varphi+\frac{\sigma_x-\sigma_y}{2\tau_*}\right)+\frac{1}{2}\psi_{,\sigma}\dot{\sigma},\qquad \lambda\geqslant 0,\\ \varepsilon_y=\frac{\lambda}{2}\left(\sin\varphi-\frac{\sigma_x-\sigma_y}{2\tau_*}\right)+\frac{1}{2}\psi_{,\sigma}\dot{\sigma},\quad \psi_{,\sigma}=d\psi/d\sigma,\\ \gamma_{xy}=\lambda\frac{\tau_{xy}}{\tau_*},\quad \dot{\sigma}=d\sigma/dt,\end{gathered}\tag{1.1}$$

where ε_x, ε_y, and γ_{xy} are the strain rates; t is time.

The relations (1.1) imply the following expression for the rate of the change in volume:

$$\varepsilon_x+\varepsilon_y=\lambda\sin\varphi+\psi_{,\sigma}\dot{\sigma}.\tag{1.2}$$

The first term on the right-hand side of (1.2) is of dilatational origin, while the second one is due to pressure change at the body points.

§2. For simulations of the body (1.1) in the three-dimensional case, a general investigation of the system of equations for plastic flow was carried out in [3]. It was established that the characteristic manifolds for the stress fields and for the rates are the same, the plastic compressibility $e_{ii}=\psi(\sigma)$ showing no effect on the field of the characteristics and only resulting in modifying the characteristic relations.

The system (1.1), (1.2) is hyperbolic and has two families of characteristics:

$$dy/dx=\operatorname{tg}\theta,\ dy/dx=-\operatorname{ctg}(\theta+\varphi),\tag{2.1}$$

where θ is the slope angle of the first characteristic (the slipping line) lying between the directions of the principal stresses ($\sigma_2>\sigma_1$) to the x axis.

In view of the results of [2], the relations (1.1) can be rewritten as

$$\varepsilon_x=\frac{\lambda}{2}\{\sin\varphi-\sin(\theta+\varphi)\}+\frac{1}{2}\psi_{,\sigma}\dot{\sigma},$$

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$$\varepsilon_y = \frac{\lambda}{2}\{\sin\varphi + \sin(\theta+\varphi)\} + \frac{1}{2}\psi_{,\sigma}\dot{\sigma},$$
$$\gamma_{xy} = \lambda\cos(\theta+\varphi). \tag{2.2}$$

We shall limit our considerations to the case of stable motion of the medium in which the pressure change at the body points is due to translation, that is,

$$\dot{\sigma} = \sigma_{,x}\, u + \sigma_{,y} v. \tag{2.3}$$

In the above, u and v are displacement rates along the Cartesian axes x and y.

There is a relation between the orthogonal projections of the velocity vector on the first and the second sliding line v_1, v_2 and the components u and v, namely,

$$u = \{v_1\cos(\theta+\varphi) - v_2\sin\theta\}/\cos\varphi,$$
$$v = \{v_1\sin(\theta+\varphi) + v_2\cos\theta\}/\cos\varphi. \tag{2.4}$$

The relations (2.2) imply that

$$\left(u_{,x} - \frac{1}{2}\psi_{,\sigma}\dot{\sigma}\right)_{\theta=0} = \left(u_{,x} - \frac{1}{2}\psi_{,\sigma}\dot{\sigma}\right)_{\theta=-\left(\frac{\pi}{2}+\varphi\right)} = 0, \tag{2.5}$$

which expresses the vanishing of the incomplete tension rate along the sliding line.

Let ds_1 and ds_2 be the arc elements of the first and the second characteristics, respectively; then one has

$$ds_1 = dx\cos\theta + dy\sin\theta,$$
$$ds_2 = -dx\sin(\theta+\varphi) + dy\cos(\theta+\varphi). \tag{2.6}$$

By using the relations (2.1), (2.3), (2.4), and (2.6), one can obtain from (2.5) the equations along the characteristic directions:

$$dv_1 - (v_1\,\mathrm{tg}\,\varphi + v_2\,\mathrm{sc}\,\varphi)d\theta + (1/2)\psi_{,\sigma}v_2\csc\varphi d\sigma = 0,$$
$$dv_2 + (v_1\,\mathrm{sc}\,\varphi + v_2\,\mathrm{tg}\,\varphi)d\theta + (1/2)\psi_{,\sigma}v_1\csc\varphi d\sigma = 0. \tag{2.7}$$

The Ketter differential equations are satisfied along the characteristic lines [2]:

$$d\sigma - 2\mathrm{tg}\,\varphi\,(c\,\mathrm{ctg}\,\varphi - \sigma)d\theta = 0,$$
$$d\sigma + 2\mathrm{tg}\,\varphi(c\,\mathrm{ctg}\,\varphi - \sigma)d\theta = 0. \tag{2.8}$$

Use is now made of the equality of the characteristics of the fields of stresses and of velocities. When (2.8) is used, the relations (2.7) become

$$dv_1 - \{v_1\,\mathrm{tg}\,\varphi + v_2\,\mathrm{sc}\,\varphi - \psi_{,\sigma}\,v_2\,\mathrm{sc}\,\varphi(c\,\mathrm{ctg}\,\varphi - \sigma)\}d\theta = 0,$$
$$dv_2 + \{v_1\,\mathrm{sc}\,\varphi + v_2\,\mathrm{tg}\,\varphi - \psi_{,\sigma}v_1\,\mathrm{sc}\,\varphi(c\,\mathrm{ctg}\,\varphi - \sigma)\}d\theta = 0. \tag{2.9}$$

Equations (2.9) represent mixed characteristic relations of a granular medium which is dilatatory and compressed irreversible by pressure. If $\psi_{,\sigma} = 0$, then Eqs. (2.9) are equal to the well-known relations [2] for an unboundedly dilatating soil.

By analyzing (1.2), (2.3), (2.5), and (2.9), one can arrive at the following conclusion: Irreversible compression due to pressure in a granular dilatatory medium results in plastic compressibility of the thin zones adjacent to the curvilinear characteristics.

It is noted that in contrast to the body simulation in [1], which was analyzed in [2] for an additively compressible medium, it is now not possible to solve the characteristic problem with known sliding lines and specified boundary conditions for the velocities once the determination stage of the plastic stress field has been passed. Of course, to be able to integrate the mixed conditions (2.9), one has to have in advance the solution of the statistically determined system (if the boundary conditions for stresses are sufficient), that is, to have a grid of sliding lines as well as the integrals for Eqs. (2.8). In [4], a general formulation of flow problems for a granular medium is given; the latter admit statically determined solutions.

§3. The case of integrable equations (2.9) is now considered [2]. For straight-line sliding curves which intersect at the angle $\pi/2 + \varphi$ the relations (2.9) are identical with the already investigated characteristic equations [2]. If one family of the sliding lines (the first family) is a beam of straight lines, and the other forms a system of logarithmic spirals (the radial shift zone), (2.9) implies that in the case of a given grid of characteristic lines and a given function $\sigma = \psi_0(\theta)$ along the second family one has $v_1 = f_1(\theta)$ along the first family and

$$v_2 \exp(\theta \operatorname{tg} \varphi) = -\operatorname{sc} \varphi \int_{\theta_0}^{\theta} f_1(\theta)\{1 - \psi_{,\sigma}(c \operatorname{ctg} \varphi - \psi_0)\} \exp(-\theta \operatorname{tg} \varphi)\, d\theta + f_2$$

along the second family of characteristics.

The function f_2 is determined from the boundary conditions and on f_1 and f_2 one imposes the condition that the dilatatory part of the rate of volume change is positive, that is, $u_{,x} + v_{,y} - \psi_{,\sigma}(\sigma_{,x}u + \sigma_{,y}v) > 0$ in the entire domain of plastic flow.

To find the distribution of the density $\rho(x, y)$ in plastic regions of the body, one writes down the continuity equation for the medium in the case of stationary motion

$$\rho_{,x}u + \rho_{,y}v + \rho(u_{,x} + v_{,y}) = 0 \tag{3.1}$$

along the characteristic lines.

From (3.1), together with (2.4)-(2.6), along the first and second family of the characteristics, respectively, one obtains the relations

$$\begin{aligned} d\rho v_2 + \rho\{dv_1 \sin \varphi(1 - \cos \varphi) - dv_2 \cos \varphi - [v_2 \operatorname{tg} \varphi - \cos \varphi(1 - \cos \varphi)v_1]d\theta\} &= 0, \\ d\rho v_1 + \rho\{dv_2 \sin \varphi(1 - \cos \varphi) - dv_1 \cos \varphi + [v_1 \operatorname{tg} \varphi - \cos \varphi(1 - \cos \varphi)v_2]d\theta\} &= 0. \end{aligned} \tag{3.2}$$

The above equations can be integrated if the geometry of the sliding lines is known as well as the velocity components v_1 and v_2 determined from the solution of (2.9). The integration constants are determined from the boundary conditions for the density on each of its characteristics.

The relations obtained by solving Eqs. (2.8), (2.9), and (3.2) for an adopted geometry of the sliding lines for the field σ, v_1, v_2, ρ are the complete solution of the mixed characteristic problem for two-dimensional flow of a granular compressible medium.

The relations which are valid along the characteristics (2.9), (3.2) can be employed to analyze the flow of granulated (powderlike) media under stresses which admit an irreversible deformation of their constituent grains, including their disintegration by melting. The latter is of paramount importance for the majority of technological processes of plastic processing by pressure exerted on nondense media, such as moulding, rolling, hydroextrusion, etc.

§4. In [1, 2] the possibility of discontinuity in the velocities of a body model [1] was investigated, and the structure of the transition layer of intensive strain rates was considered. It was established that the transition layer is of finite thickness; the normal velocity component suffers a jump during the transition across the discontinuity line (of the characteristic), the change in the velocity vector during the transition across the discontinuity layer making an angle φ with the latter.

The discontinuities in the velocities are now considered for the relations (2.2). Let the axes of x, y be along the tangent and normal to the midline of the transition layer; then, as implied by (2.2), to satisfy the relations $u_{,y}$, $v_{,y} \gg u_{,x}$, $v_{,x}$ it is necessary that the transition layer adjoins the characteristic $(dy/dx = 0)$ along which the partial deformation rate vanishes in accordance with (2.5).

In this case it follows from (2.2) that

$$u_{,x} = \frac{1}{2} \psi_{,\sigma}\dot{\sigma}, \quad v_{,y} = \lambda \sin \varphi + \frac{1}{2} \psi_{,\sigma}\dot{\sigma}, \quad u_{,y} = \lambda \cos \varphi. \tag{4.1}$$

For $dy/dx = 0$ the relations (4.1) imply that on the discontinuity line one has

$$\frac{[v]}{[u]} = \operatorname{tg} \varphi + \frac{\psi_{,\sigma}\dot{\sigma}}{\lambda \cos \varphi}, \quad [u] = u^+ - u^-, \quad [v] = v^+ - v^-. \tag{4.2}$$

For the simulation under consideration the transition layer must be of finite thickness δ ($\delta \neq 0$, since $[v] \neq 0$). Indeed, for continuously increasing δ and for finite jumps $[u]$, $[v]$ the functions ε_y, γ_{xy}, λ will decrease without bound; in this case it follows from (4.2) that resistance to hydrostatic compression in the layer material must increase without bound, which is not feasible from the physics point of view.

In [1], "simple discontinuous" solutions were introduced based on the expression for the rate of energy dissipation on a unit area of disruption:

$$W = c[u]. \tag{4.3}$$

Using the relation (4.2) and the results in [5], one finds that the formula (4.3) for an additively compressible medium assumes the form

$$W = (c \cos \varphi - \sigma \sin \varphi)\{[u]^2 + [v]^2\}^{1/2} + \sigma[v].$$

By employing theorems on extremals the latter expression can be used to find upper bounds for limit loads in the case of motion of granular materials subjected to high pressures.

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DYNAMIC DEFLECTION OF STIFFLY PLASTIC RESTRAINED CIRCULAR PLATES WITH THE EFFECTS OF SHEAR AND ROTATIONAL INERTIA TAKEN INTO ACCOUNT

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Upon the action of transient dynamic loads of high intensity, a large part of the internal energy supplied to a structure can be dissipated into the work of plastic deformations prior to the structure being destroyed or receiving inadmissible residual deformations. Approaches based on the model of a stiffly plastic body have found wide application for the solution of the corresponding problems based on estimation of the extent of damage to structures from the action of "explosive" loads. A detailed review of Soviet and foreign research conducted in recent years in this area is given in [1]. Experimental investigations conducted in a number of papers [2-4] have revealed that the values of the residual deflections and rotation angles measured in the experiments turn out to be appreciably less than the theoretical values, amounting to approximately 20-80% of them. This discrepancy is explained by the effect of a number of factors which are not taken into account in the theory mentioned above. In particular, the effect of rotational inertia and the limitedness of resistance to transverse shears is not taken into consideration in all investigations known up to now. A theory of the dynamic deflection of circular plates made of a stiffly plastic material is developed in this paper which takes account of rotational inertia and the reduced resistance to transverse shear. It is shown that the nature of the dynamic behavior and the energy dissipation mechanism in the case of plastic deformations is significantly different in this case than in the theories mentioned above.

§1. We will consider the axisymmetrical deformation of a circular plate made of an ideally stiffly plastic material. We will assume the following kinematic hypotheses of the deformation of the plate (in the cylindrical coordinate system r, φ, z tied to the median surface):

$$u_r = zu(r, t),\ u_\varphi \equiv 0,\ u_z = W(r, t),$$

where u_r, u_φ, and u_z are the components of the displacement vector and t is the time. Using Lagrange's variational principle in combination with the d'Alembert principle, we obtain, respectively, the equations of motion of the plate

$$(1/3)\ddot{u} = m_1' + x^{-1}(m_1 - m_2) - ql + \varphi, \quad \ddot{w} = q' + x^{-1}q + p \tag{1.1}$$

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